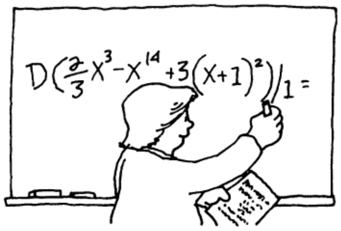
“Work is often the father of pleasure.”
Voltaire



A Solution to the Yin-Yang Equilibrium Problem[1]

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1-2 Idea of Solution

To solve the problem, we make the assumption that this plate is two-dimensional (flat) and has a uniform density. Clearly, the plate will be balanced vertically if suspended from any point lying on the line $x = \bar{x}$, where $\bar{x}$ is the x-coordinate of its center of mass (COM).

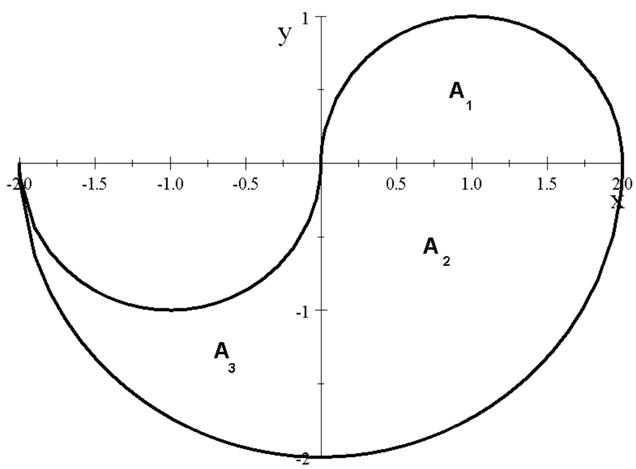

Figure 1

[1] Submitted to the Challenging Problem Section of the F5 Magazine for Undergraduate Physicists, Saudi Physical Society, First Issue, July 2008

Next we place the origin of the coordinate system at the center of the big circle (See Figure 1). The general formula for finding the x-coordinate of the COM for a region D having density ρ is

$$\bar{x} = \frac{1}{M}\iint_D x\,\rho(x,y)\,dA\,,$$

where the mass M is given by

$$M = \iint_D \rho(x,y)dA.$$

But since the density is constant, we may safely write

$$\bar{x} = \frac{1}{A}\iint_D x\,dA\,,$$

where $A = \iint_D dA$ is the area of the plate.

Now we divide the area to three regions or subareas. For the subarea A_1, the COM is clearly $\overline{x_1} = 1$ (according to the symmetry principle[2]). So if we compute $\overline{x_2}$ and $\overline{x_3}$ (the centers of masses of A_2 and A_3 respectively), then we will have like three points with their masses. This enables us to apply the basic formula

$$\bar{x} = \frac{m_1\overline{x_1} + m_2\overline{x_2} + m_3\overline{x_3}}{M}\,, \qquad (1)$$

where M is the total mass of the plate & m's the partial masses, to obtain the x-coordinate of the COM of the plate.

2-2 Calculations

In calculating the double integrals involved in the COM formula, we best describe D in polar coordinates. In $\overline{x_2}$ we have

$$D = \{(r,\theta)\,|\,0 \le r \le 2,\ 0 \le \theta \le -\pi/2\}$$

The subarea A_3 is bounded by the circle $(x+1)^2 + y^2 = 1$, which becomes in polar coordinates $r = -2\cos\theta$, and then we have

$$D = \{(r,\theta)\,|\,-2\cos\theta \le r \le 2,\ \pi \le \theta \le 3\pi/2\}.$$

[2] which says that if a region in plane is symmetric about a line, then its COM must lie on that line.

$$\overline{x_2} = \frac{1}{A_2}\iint_D x\,dA$$

$$= \frac{1}{\pi}\int_0^2 \int_{-\pi/2}^{0} r\cos\theta(r d\theta\, dr) = \frac{1}{\pi}\int_0^2 \int_{-\pi/2}^{0} r^2 \cos\theta\, d\theta\, dr$$

$$= \frac{1}{\pi}\ \sin\theta\Big|_{-\pi/2}^{0}\ \frac{r^3}{3}\Big|_0^2 = \frac{1}{3\pi}(0+1)(8-0) = \frac{8}{3\pi} \approx 0.849$$

$$\overline{x_3} = \frac{1}{A_3}\iint_{D_3} x\,dA = \frac{2}{\pi}\int_{\pi}^{3\pi/2}\int_{-2\cos\theta}^{2} r^2\cos\theta\ dr d\theta$$

$$= \frac{2}{\pi}\int_{\pi}^{3\pi/2}\left(\frac{r^3}{3}\Big|_{-2\cos\theta}^{2} \cos\theta\right) d\theta = \frac{2}{3\pi}\int_{\pi}^{3\pi/2}\left(8+8\cos^3\theta\right)\cos\theta\ d\theta$$

$$= \frac{16}{3\pi}\int_{\pi}^{3\pi/2}\left(1+\cos^3\theta\right)\cos\theta\ d\theta = \frac{16}{3\pi}\int_{\pi}^{3\pi/2}\left(\cos\theta+\cos^4\theta\right) d\theta$$

$$= \frac{16}{3\pi}\left[\left(\sin\theta+\frac{1}{4}\cos^3\theta\sin\theta\right)\Big|_{\pi}^{3\pi/2} + \frac{3}{4}\left(\frac{1}{2}\theta+\frac{1}{4}\sin 2\theta\right)\Big|_{\pi}^{3\pi/2}\right]$$

$$= \frac{16}{3\pi}\left[(-1-0)+\frac{1}{4}(0)+\frac{3}{(4)(2)}\frac{\pi}{2}+\frac{3}{12}(0)\right]$$

$$= \frac{16}{3\pi}\left[-1+\frac{3\pi}{16}\right] = 1-\frac{16}{3\pi} \approx -0.697$$

Now, substituting into (1),

$$\bar{x} = \frac{\pi\left(0.5+0.849-0.697/2\right)}{2\pi}$$

$$= 0.5R$$

where R is the radius of the smaller circle. Of course, m is equal to “area times density”, but again since density is constant, it is canceled out with its analogy in the denominator. Thus, for a plate with $R = 50\,mm$, the plate would be balanced vertically if suspended at $x = 25\,mm$. It is interesting to make a plate like this by hard paper and handle it from the line x=0.5 R… And enjoy visualizing your work by experiment.