Ocean Surface Dynamics

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December 2009

1 Abstract

In this paper, we model the movement of ocean's surface when a wind blows over it. After describing and formulating the accelerations and forces acting on an arbitrary point of ocean's surface, we arrive at the Navier-Stokes equation for a rotating fluid. We write its component forms and reduce them to obtain a second order differential equation with constant coefficients, called the steady Ekman equation. To solve it, we develop two boundary conditions. Finally, we explore several properties of the solution of this equation and other related equations.

2 Introduction

The ocean, covering nearly 71% of Earth's surface and affecting our daily life in many ways, has led curious people throughout history and around the globe to explore it in different aspects for different reasons. One of the most important and vital aspects is its integral role in Earth's climate and weather systems. Clearly, ocean transports heat, received from the Sun, by means of currents.This in turn leads to variations in weather patterns in short term (e.g. El Nino phenomenon and the associated changes in weather patterns), and to climate changes in the long term. Applied mathematicians, equiped with exploring tools, have been involved cooperatively with scientists and explorers to study the movements of ocean water. This paper highlights a historical example in which an analytic solution of differential equations was used to analyze successfully a real situation. The goal is to urge math students to consider areas of job or research related to meteorology or oceanography which are filled with intellectual challenges for them and their solutions would contribute to humanity.

In 1893-1896, and during the Fram expedition, the Norwegian explorer Nansen noticed that wind tended to blow ice at an angle to the right of and not in the same direction of the blowing wind. He then asked Vilhelm Bjerknes, a Norwegian professor of fluid dynamics, to let one of his students make a theoretical study of the influence of Earth's rotation on wind-driven currents. Bjerknes invited Walfrid Ekman and he presented his findings in his doctoral thesis in 1902. Ekman's work was "the first of a remarkable series of studies conducted during the first half of the twentieth century that led to an understanding of how winds drive the ocean's circulation" [3]. In this paper, we consider the mathematical model that Ekman developed to explain the angle at which surface currents make with a steady blowing wind.

3 The Modeling

To model the movement of ocean's surface means to write its equation of motion, Newton's second low ($F = ma$), which describes the velocity of surface currents. To specify the forces and accelerations involved, we need to specify a frame of reference. Taking an arbitrary point of ocean's surface as an origin, we set a three-dimensional Cartesian coordinate system where the x-axis points east, the y-axis points north, and the z-axis points vertically up. This frame of reference is attached to the Earth, which rotates around its North-South axis at constant angular speed ω. This means our chosen coordinate system is accelerating because its origin constantly changes its direction of velocity. This non-inertial frame of reference is attached to, we assume, an inertial frame of reference, a Cartesian coordiante system attached to the center of the Earth. This assumption is based on the fact that the orbital motion of Earth around the sun has such a large radius of curvature that it is straight (hence, non-accelerating) to a very good approximation.

3.1 Accelerations

The equation of the acceleration of a particle in the rotating system with respect to the inertial frame of reference is

$$\frac{du}{dt} = \frac{du'}{dt} + \dot{\omega} \times r' + 2\omega \times u' + \omega \times (\omega \times r') + \omega \times \omega \times R,$$

where $\frac{du'}{dt}$ is the acceleration observed in the rotating system. The term $\dot{\omega} \times r'$, called the transverse acceleration, appears if the angular velocity vector is changing in either magnitude or direction or both. The term $2\omega \times u'$, known as the Coriolis acceleration, appears whenever a particle moves in a rotating coordinate system (except when the velocity u' is parallel to the axis of rotation). The coriolis acceleration deflects moving particles at right angles to its direction of motion. The term $\omega \times (\omega \times u')$, which is called the centripetal acceleration, is directed toward the axis of rotation and is perpendicular to the axis. We see that the transverse term is zero, and ignore the centripetal accelerations because their values are small compared to gravity.

The velocity of the current is a function of x, y, z, and t, and it consists of an Eastern component, Northern component, and vertical component:

$$(u, v, w) = u = u(x, y, z, t).$$

In our model we are interested only with the horizontal movement of the current and so we neglect the vertical component, w.

The term $\frac{du'}{dt}$ describes the parcel acceleration in the rotating frame of reference; In other words, it describes the change in velocity. When we talk about change in quantity in a fluid, there are two views describing this change. We could describe how the velocity changes with respect to an arbitrary fixed location (position) within the fluid, or, another view, we can describe how the velocity of an arbitrary parcel is changed with time, tracing the trajectory of a particular parcel. The first is called Eulerian and the second is called Lagrangian. The velocity vector depends on position and time. Hence, the total derivative of u with respect to t is

$$\begin{aligned}\frac{du}{dt} &= \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt} + \frac{\partial u}{\partial t} \\ &= \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right) \cdot \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) + \frac{\partial u}{\partial t} \\ &= (u.\nabla)u + \frac{\partial u}{\partial t}\end{aligned}$$

The total derivative is the Lagrangian rate of change while the partial derivative is the Eulerian rate of change. The difference is what we call, in oceanographic and meteorological contexts, advective acceleration. The Eulerian rate of change represents the local rate of change that is measured by a fixed observer's location. The advective acceleration is the advective change that occurs due to fluid being transported into a new location. The total change, which is experienced by an observer moving with the speed of the fluid, is produced by the sum of these two changes. Thus, mathematically, in the language of differential calculus, the total derivative allow all independent variables to vary, while the local derivative with respect to t holds x, y, and z constants [2].

The Coriolis acceleration is

$$\begin{aligned} 2\omega \times u' &= 2 \begin{vmatrix} i & j & k \\ 0 & \omega\cos\theta & \omega\sin\theta \\ u & v & w \end{vmatrix} \\ &= 2\left(\omega w\cos\theta - \omega v\sin\theta, \omega u\sin\theta, -\omega u\cos\theta\right) \end{aligned}$$

Because $w \ll v$, the $\omega w\cos\theta$ term can be safely neglected. Hence, we express the Coriolis acceleration here, to a good degree of accuracy, as $(-fv, fu, 0)$ where $f = 2\omega\sin\theta$, called the Coriolis parameter.

3.2 Acting Forces

We consider the forces acting on a parcel of water in ocean surface. There are three forces acting in this situation, the first of which is of course gravity. Second, there are pressure forces acting accross opposite faces of the cube. The presense of pressure differences transfers momentum; the gradient of pressure is the force that acts on the cube. Horizontal pressure gradients act to transfer momentum from regions of high pressure to regions of low pressure. So we have, using Gauss theorem,

$$\begin{aligned} -\int_V \rho g\overline{k}\, dV + \int_S (-p\overline{n})\, dS &= -\int_V \rho g\overline{k}\, dV - \int_V \nabla p\, dV \\ &= \int_V (-\rho g\overline{k} - \nabla p) dV. \end{aligned}$$

The third force is friction. Friction here occurs in water surface due to its contact with moving air (steady wind). That is, the force that occurs when a parcel moves past another parcel. Water is a Newtonian fluid; that is, it has a linear relationship between any applied shear and the vertical velocity gradient. The constant of proportionality is called the dynamic viscosity of the fluid. Its value indicates how resistant the fluid is in response to shearing (deformation). The effect of viscous shearing extends just over a few millimeters. We would add these viscous stresses to the equation of motion as friction. In our situation, however, the steady wind sets surface water in motion, called currents. The transfer of momentum starting from the very surface layer which touches the moving air to the last affected layer occurs through small eddies. This is called turbulent flow and it happens when the velocity is high. In other words, because the velocity of this powerful wind is high, every layer slides past the lower layer through turbulence. The whole layer over which the velocity changes due to the blowing wind is called a boundary layer. We assume the same linear relationship between the turbulent viscous shear and the vertical velocity gradient. The constant of proportionality, which is very larger than the dynamic viscosity, is called the eddy viscosity, v_v.

Now, we can write the equation of motion for the ocean

$$\frac{\partial u}{\partial t} + (u.\nabla)u + 2\omega \times u = -\frac{1}{\rho}\nabla p + g + friction, \tag{1}$$

which is Navier-Stokes equation for a rotating fluid.

3.3 Formulation of Friction

In order to see how these turbulent viscous stresses arise from the equation of motion, we consider the mean flow over a short time interval, and write every current as

$$u = \overline{u} + u' \tag{2}$$

where $\overline{u}$ is the mean flow and u' is the difference between the exact value and the mean flow. We then substitute (2) into (1) and take the mean value of the whole equation. Note that the mean value of u' is zero:

$$\begin{aligned} \overline{u'} &= \frac{1}{T}\int_{t-\frac{T}{2}}^{t+\frac{T}{2}} (u-\overline{u})dt = \frac{1}{T}\int_{t-\frac{T}{2}}^{t+\frac{T}{2}} u\, dt - \frac{1}{T}\int_{t-\frac{T}{2}}^{t+\frac{T}{2}} \overline{u}\, dt \\ &= \overline{u} - \frac{1}{T}\overline{u}\, t\Big|_{t-\frac{T}{2}}^{t+\frac{T}{2}} = 0 \end{aligned}$$

After simplifying, we obtain the same equation exept for the extra term

$$\overline{(u'.\nabla)u'}$$

This term is a tensor consisting of nine components which are called Renolds stresses:

$$\begin{pmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{pmatrix} = \begin{pmatrix} -\rho\overline{u'^2} & -\rho\overline{u'v'} & -\rho\overline{u'w'} \\ -\rho\overline{v'u'} & -\rho\overline{v'^2} & -\rho\overline{v'w'} \\ -\rho\overline{w'u'} & -\rho\overline{w'v'} & -\rho\overline{w'^2} \end{pmatrix}$$

The terms in the first row, e.g., transfer eastward momentum in the x, y, and z directions. Thus, the frictional force in the x-direction would be

$$F_x = \frac{\partial \tau_{11}}{\partial x} + \frac{\partial \tau_{12}}{\partial y} + \frac{\partial \tau_{13}}{\partial z}$$

"Each τ_{ij} measures the covariance between two of the fluctuating components $u' = (u', v', w')$."[1]. As we stated above, because these terms play the same role in transfering momentum that viscous stresses play, we assume these terms have the same linear relationship; i.e,

$$\tau_{31} = -\rho\overline{w'u'} = -\rho v_v \frac{\partial u}{\partial z},$$

and if v_v is constant, then

$$\frac{\partial}{\partial z}\tau_{31} = v_v \frac{\partial^2 u}{\partial z^2}.$$

Now we are ready to write the x and y component form of the vector equation (1):

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x} + v_v\frac{\partial^2 u}{\partial z^2} + v_H\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y} + v_v\frac{\partial^2 v}{\partial z^2} + v_H\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

In the next section, we make some assumptions to reduce these two equations to the steady Ekman equations and solve them.

4 The Solution

If we are interested only in the vertical transfer of momentum, then we neglect the terms containing v_H, eddy viscosity of horizontal transfer. Because we ignored the vertical component of the velocity of the current, terms containing w are zero. Ekman also assumed that the horizontal flow is constant and steady, which means

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial x} = \frac{\partial}{\partial y} = 0$$

The equations thus reduce to

$$\begin{aligned} -fv &= -\frac{1}{\rho}\frac{\partial p}{\partial x} + v_v \frac{\partial^2 u}{\partial z^2} \\ fu &= -\frac{1}{\rho}\frac{\partial p}{\partial y} + v_v \frac{\partial^2 v}{\partial z^2} \end{aligned}$$

In deep ocean, the velocity of currents is no longer affected by any external forces, and the equation of motion (1) reduce to

$$2\omega \times u = -\frac{1}{\rho}\nabla p + g,$$

or, in component form

$$\begin{aligned} \frac{\partial p}{\partial x} &= \rho f v \\ \frac{\partial p}{\partial y} &= -\rho f u \\ \frac{\partial p}{\partial z} &= -\rho g \end{aligned} \tag{3}$$

These equations represent geostrophic flow, in which the coriolis force is balanced with the horizontal pressure gradient and gravity with the vertical pressure. When this happens, deep water moves horizontally in circular patterns over constant pressure levels, see Figure 1.

Then if we write

$$(u, v) = (u_G + u_E, v_G + v_E),$$

such that (u_G, v_G) , the geostrophic part, remains independent of z and t, then, using (3), we have

$$
\begin{aligned}
-fv_E &= v_v \frac{\partial^2 u_E}{\partial z^2} \\
fu_E &= v_v \frac{\partial^2 v_E}{\partial z^2}
\end{aligned}
$$

which are the steady Ekman equations. There are two methods to solve them.

4.1 Method One

Using substitution, we obtain a fourth-order linear differential equation with constant coefficients

$$\frac{\partial^4 u_E}{\partial z^4} + \left(\frac{f}{v_v}\right)^2 u_E = 0.$$

To solve it, we write the associated auxiliary equation

$$r^4 + \left(\frac{f}{v_v}\right)^2 = 0,$$

which has roots

$$r^2 = \pm i \frac{f}{v_v} \implies r = \pm\sqrt{\frac{if}{v_v}}$$

But using De Moivre's Theorem, we write r as a standard complex number to be able to write the general solution

$$r = \cos\frac{\pi}{4} + i \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$$

Then

$$r = \pm\sqrt{\frac{f}{2v_v}}(1+i) = \pm\delta(1+i),$$

where we set $\delta = \sqrt{\frac{f}{2v_v}}$.
Then the general solution is

$$
\begin{aligned}
u_E &= Ae^{\delta(1+i)z} + Be^{-\delta(1+i)z} \\
v_E &= Ce^{\delta(1+i)z} + De^{-\delta(1+i)z}
\end{aligned}
$$

The first boundary condition is clearly

$$u_E, v_E \to 0 \text{ as } z \to -\infty,$$

which makes $B = D = 0$. To obtain A and C, we substitute the general solutions into the original equations, as follows

$$fu_E = v_v \frac{\partial^2 v_E}{\partial z^2} \Longrightarrow fAe^{(1+i)\delta z} = v_v \frac{\partial^2}{\partial z^2} Ce^{(1+i)\delta z} = v_v Ce^{(1+i)\delta z}(1+i)^2\delta^2$$

Then $A = iC$

The second boundary condition, at $z = 0$, we have $\tau = \tau_{xz} = v_v \frac{\partial u_E}{\partial z}$, and then we have

$$\begin{aligned} v_v \left.\frac{\partial u}{\partial z}\right|_{z=0} &= \tau \Longrightarrow v_v A(1+i)\delta = \tau \text{ and} \\ A &= \frac{\tau}{v_v(1+i)\delta} = \frac{\tau(1-i)}{2v_v\delta} \end{aligned}$$

Then, we obtain

$$\begin{aligned} u_E &= \tau\sqrt{\frac{1}{fv_v}} e^{\delta z} \cos\left(-\delta z + \frac{\pi}{4}\right) \\ v_E &= -\tau\sqrt{\frac{1}{fv_v}} e^{\delta z} \sin\left(-\delta z + \frac{\pi}{4}\right) \end{aligned}$$

for $z \leq 0$. The details of this solution is given in [5].

4.2 Method Two

If we add the first equation to i times the second equation, we obtain a second order differential equation with constant coefficients

$$ifW = v_v \frac{\partial^2 W}{\partial z^2}$$

We have to develop two boundary conditions. One, as we get deeper into the sea, W gets smaller and smaller, i.e,

$$W \to 0 \text{ as } z \to -\infty.$$

Two, at the sea surface ($z = 0$), we have

$$\tau = (\tau^x, \tau^y) = -\rho v_v \left(\frac{\partial u_E}{\partial z}, \frac{\partial v_E}{\partial z} \right) = -\rho v_v \frac{\partial W}{\partial z},$$

where $\tau = (\tau^x, \tau^y) = \tau^x + i\tau^y$ is the horizontal wind stress vector. The the solution is

$$W = -\frac{\tau}{\rho\sqrt{if v_v}} \exp\left\{ z\sqrt{\frac{if}{v_v}} \right\}$$

5 Analysis

At the sea surface ($z = 0$), the current moves to the right of the wind in the Northern hemisphere at an angle of 45°.Every successive layer then moves slightly slower to the right of the above layer until the coriolis force completely balances the horizontal pressure gradient, the geostrophic flow. This movement of surface currents is called Ekman Spiral, see Figure 2. The layer over which the velocity changes from that of the wind to the unaffected region by the wind is called boundary layer. Here, it is named Ekman layer, and is usually around hundred meters thick. The reason why the movement is to the right of the wind and not to the left is as follows: It actually is to the right of the wind in the Northern hemisphere and to the left of the wind in the Southern hemisphere, and this is due the direction of the coriolis force, which results from the cross product the velocity and the angular velocity of Earth, counterclockwise or clockwise.

6 References

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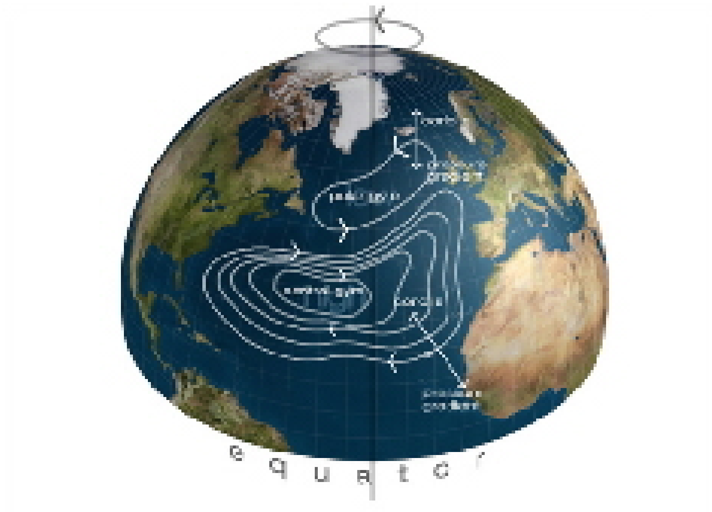

Figure 1: Geostrophic Flow

Aknowledgement

I would like to thank Dr. Najwa Jawharji for technical help on complex numbers.

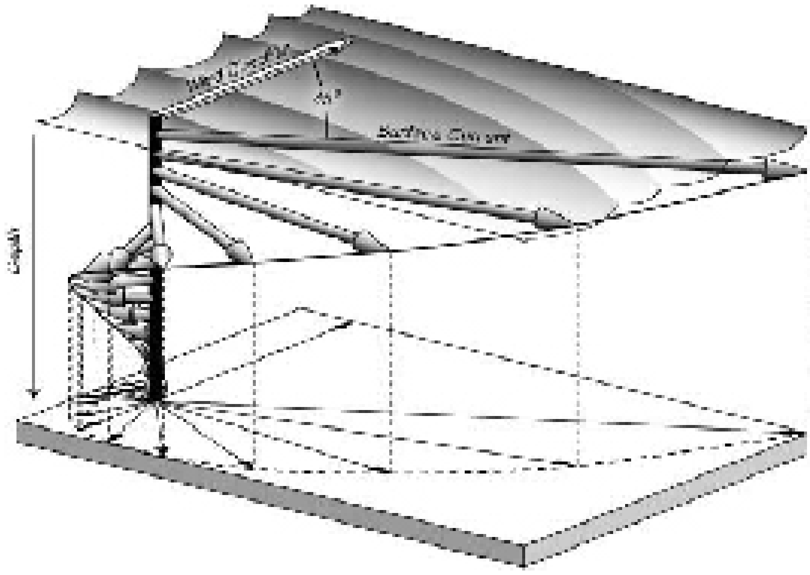

Figure 2: Ekman Spiral